e-ISSN: 2149-4584

Erzincan University Journal of Science and Technology

EJSAT 2025, 18 (3) 735-744

Research Article

On a Conjecture about Profiles of Finite Connected Racks

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Received: 26/10/2024, **Revised:** 29/07/2025, **Accepted:** 29/07/2025, **Published:** 31/12/2025

Abstract

A rack is a set equipped with a binary operation for which all left multiplications are automorphisms. A quandle is a rack that satisfies an additional axiom. In a finite connected rack, the cycle type of the permutation induced by left multiplication is independent of the chosen element, and is called the profile of the rack. Hayashi conjectured that, in the profile of a finite connected quandle, the length of any cycle divides the length of the longest cycle. In this paper, we prove Hayashi's conjecture in several specific cases.

Keywords: Finite connected rack, profile of a rack, Hayashi's conjecture.

Sonlu Bağlantılı Rakların Profilleri Hakkında Bir Sanı Üzerine

Öz

Bir rak bir küme ve üzerinde tanımlanmış ikili işlemden oluşup sol çarpımlar kümenin otomorfizmalarıdır. Bir kuandle ise ikili işlemin belli bir aksiyomu sağladığı rak olarak tanımlanır. Sonlu bağlantılı bir rak için bir eleman ile sol çarpımın belirlediği permütasyonun döngü tipi seçilen elemandan bağımsızdır ve bu döngü tipine rakın profili denilir. Hayashi sonlu bağlantılı bir kuandlenin profilinde herhangi bir döngünün uzunluğunun en uzun döngünün uzunluğunu böldüğünü öne sürmüştür. Bu makalede Hayashi'nin sanısını bazı belli durumlar için ispatlıyoruz.

Anahtar Kelimeler: Sonlu bağlantılı rak, rakın profili, Hayashi'nin sanısı.

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Cite this Article: Erzincan University Journal of Science and Technology 2025, 18(3)735-744. https://doi.org/10.18185/erzifbed.1591413

1. Introduction

Racks and quandles are algebraic objects that are mostly studied in the context of knot theory. The defining properties of those objects are in a sense compatible with the Reidemeister moves which makes them useful to define knot invariants. For example, in [1] Joyce introduced *knot quandle* and showed that it is a complete invariant for knots. This result was also obtained in [2] by Matveev independently in which the author introduced the notion of distributive groupoid. Also, in [3] Fenn and Rourke proved that the *fundamental rack* is a complete invariant for irreducible framed links in a 3-manifold. There are various notational conventions used to define racks and quandles in literature. In this paper we define racks and quandles in the following way.

A *rack* X is a set together with a binary operation $\triangleright: X \times X \to X$ satisfying the following two axioms:

- (A1) for all $x, y, z \in X$ we have $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$
- (A2) for all $x, z \in X$ there is a unique $y \in X$ such that $x \triangleright y = z$

A rack X is a *quandle* if, additionally, it satisfies the following axiom:

- (A3) for all $x \in X$ we have $x \triangleright x = x$

Let G be a group and for any two elements $a, b \in G$ let $a \triangleright b := aba^{-1}$. Then, the group G as well as any conjugacy class of G satisfies those three axioms; hence equipped with the conjugation operation $\triangleright$ each one of them is an example of a quandle. More generally, we call a subset X of G *conjugation rack* if it is a rack with the conjugation operation inherited by the group.

In this paper we only consider finite racks, although some of the statements are also valid for infinite racks. It is customary to identify elements of a finite rack X with integers $1,2, \dots, n$ if X has n elements. By axiom (A2), the map $\phi_x: y \to x \triangleright y$ is a permutation hence is an element of the symmetric group S_n of degree n. A permutation $\sigma \in S_n$ is an automorphism of X if and only if $\sigma(x \triangleright y) = \sigma(x) \triangleright \sigma(y)$ for every $x, y \in X$. By axiom (A1), the map ϕ_x is an automorphism of X.

The set of all automorphisms of X, denoted $\mathsf{Aut}(X)$, form a subgroup of S_n which is called the *automorphism group* of X. The map $\Phi: x \mapsto \phi_x$ defines a rack morphism from X to S_n, where S_n is considered as a quandle with conjugation operation (see Proposition 2). The *inner automorphism group* of X, denoted $\mathsf{Inn}(X)$, is the subgroup of $\mathsf{Aut}(X)$ generated by the elements of $\Phi(X)$. Let σ be an automorphism of X. Since $\sigma\phi_x\sigma^{-1} = \phi_{\sigma(x)}$ for every $x \in X$, we see that $\mathsf{Inn}(X)$ is a normal subgroup of $\mathsf{Aut}(X)$. In general $\mathsf{Inn}(X)$ and $\mathsf{Aut}(X)$ may be different groups.

A rack X is *faithful* if the map $\Phi: X \to \mathsf{Inn}(X)$ is injective. Suppose X is faithful. Obviously, the map Φ is an isomorphism between X and $\Phi(X)$, hence X is a quandle. Moreover, the center of

the inner automorphism group is trivial and the action of $\mathsf{Inn}(X)$ on $\Phi(X)$ by conjugation is faithful (see Proposition 2).

A rack X is *connected* if the action of its inner automorphism group on X itself is transitive. Suppose X is a connected rack. In that case, any two elements of $\Phi(X)$ can be conjugated by an element of $\mathsf{Inn}(X)$. Since in symmetric group conjugate elements have the same cycle type, connectedness of X implies each element of $\Phi(X)$ has the same cycle type. This cycle type $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$ is called the *profile* of X. Here λ_s $(0 \leq s \leq t)$ are ordered increasingly. Notice that $x \rhd x = x$ for every $x \in X$ if X is a quandle which means in the profile λ of a quandle X the value of the least element λ_0 must always be 1. The purpose of this paper is to prove some special cases of the following conjecture which was stated by Hayashi originally for quandles (see [4, Conjecture 1.1]).

Conjecture 1. *For a finite connected rack X with profile* $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$*, each* λ_s $(0 \leq s \leq t)$ *divides* λ_t.

Let G be a finite group, α be an automorphism of G, and H be a subgroup of G pointwise fixed by α, i.e., $\alpha(h) = h$ for every $h \in H$. One can easily verify that the coset space G/H with the operation $xH \rhd yH := x\alpha(x^{-1}y)H$ satisfies quandle axioms. It is called the *homogeneous quandle* and denoted by (G, H, α). Any finite connected quandle is isomorphic to a homogeneous quandle. Therefore, given the complete list of transitive groups of degree n, it is possible to determine the complete list of non-isomorphic connected quandles of n elements. This method is used by Vendramin to compute all connected quandles having less than 48 elements (see [5]). The list is available in Rig [6], a GAP package designed for computations related to racks and quandles.

We shall prove that Conjecture 1 holds when the inner automorphism group acts on the rack primitively (see Theorem 6). In Table 1 we present permutations of a quandle which is the fourth quandle with 12 elements in Vendramin's list. Here elements of the rack are labeled by the integers $1,2,\dots,12$ and each inner automorphism ϕ_i, for $1 \leq i \leq 12$, is represented as a permutation of the elements of the quandle. The reader may easily verify that the following partition

$$\{1,5,9\},\ \{2,6,10\},\ \{3,7,11\},\ \{4,8,12\}$$

formed by taking the first three numbers in each row of the table is a block system for the action of the inner automorphism group on the rack itself. Thus, the action of the inner automorphism group is not primitive. From the table, it is apparent that the profile of this rack is λ=(1,2,3,6), so Hayashi's conjecture holds in this case as well.

Table 1: Permutations of SmallQuandle(12,4) in Rig

ϕ_1 : (1)(5,9)(2,4,3)(6,12,7,10,8,11)
ϕ_2 : (2)(6,10)(1,3,4)(5,11,8,9,7,12)
ϕ_3 : (3)(7,11)(1,4,2)(5,12,6,9,8,10)
ϕ_4 : (4)(8,12)(1,2,3)(5,10,7,9,6,11)
ϕ_5 : (5)(1,9)(6,8,7)(2,12,3,10,4,11)
ϕ_6 : (6)(2,10)(5,7,8)(1,11,4,9,3,12)
ϕ_7 : (7)(3,11)(5,8,6)(1,12,2,9,4,10)
ϕ_8 : (8)(4,12)(5,6,7)(1,10,3,9,2,11)
ϕ_9 : (9)(1,5)(10,12,11)(2,8,3,6,4,7)
ϕ_{10} : (10)(2,6)(9,11,12)(1,7,4,5,3,8)
ϕ_{11} : (11)(3,7)(9,12,10)(1,8,2,5,4,6)
ϕ_{12} : (12)(4,8)(9,10,11)(1,6,3,5,2,7)

Let A be a finite abelian group and α be an automorphism of A. The corresponding homogeneous quandle $(A, 1, \alpha)$ is called an *affine quandle*. Let $\beta(x) := x - \alpha(x)$ for every $x \in A$. Observe that in $(A, 1, \alpha)$ the quandle operation is defined by

$$x \triangleright y := \alpha(y - x) + x = \alpha(y) + \beta(x) = \beta(x - y) + y$$

and the orbit of y under the action of inner automorphism group is $y + \beta(A)$. Suppose $(A, 1, \alpha)$ is connected. Then β must be a bijection of A. Moreover, the inner automorphism group is isomorphic to a cyclic extension of A, hence is solvable. Now, consider the profile of $(A, 1, \alpha)$. Kajiwara and Nakayama proved in [7] that there must be an element of A whose orbit size under the action of $\langle \phi_x \rangle$ is the order of ϕ_x. Let $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$ be the profile of $(A, 1, \alpha)$. Since the least common multiple of the numbers $\lambda_1, \dots, \lambda_t$ divides the order of ϕ_x, Kajiwara and Nakayama's result shows that Hayashi's Conjecture holds for connected affine quandles.

Let C be a conjugacy class of a symmetric group S_d. Hence, the inner automorphism group of C is either isomorphic to A_d or S_d. In particular $\mathsf{Inn}(C)$ is not solvable if $d \geq 5$. Let x be an element of C and let ℓ be the order of x in S_d. To prove Hayashi's Conjecture holds for C, one possible way is to find an element y of C so that $x^k y x^{-k} \neq y$ for $1 \leq k < \ell$. However, this method seems to be not applicable for other classes of groups as we don't have an explicit description of the elements of conjugacy class C.

The aim of this paper is to show that Hayashi's Conjecture holds for some particular classes of racks. The organization of the paper is as follows. In Section 2 we review some of the basic facts on racks and give a group theoretical interpretation of Conjecture 1 (see Proposition 5). In Section 3 we state and prove our main results. To be more precise in Theorem 6 we shall prove that Conjecture 1 holds when the inner automorphism group acts on the rack primitively and in

Theorem 7 we shall show that Conjecture 1 holds for the conjugacy classes in a symmetric group. In either Theorem the proof uses Proposition 5.

2. Prelimineries

The reader may refer to [8] for the basic theory of racks and in particular to Lemma 1.7, Lemma 1.9 and Proposition 3.2 of the same paper for alternative statements and proofs of the following Proposition.

Proposition 2. *Let X be a rack. Then following statements hold.*

(i) The map $\Phi: X \to Inn(X)$ defines a rack morphism between X and $\Phi(X)$ which is an isomorphism if X is faithful.

(ii) If X is faithful, then the action of $Inn(X)$ on X by automorphisms and on $\Phi(X)$ by conjugation are isomorphic and the center of $Inn(X)$ is trivial.

(iii)If X is a conjugation rack and $G := \langle X \rangle$, then the map $\Phi: X \to \Phi(X)$ extends to a map

$$\Phi: G \to Inn(X),$$

(iv) which is a group homomorphism and $Inn(X) \cong G/Z(G)$.

(v) The rack X is connected and faithful if and only if it is isomorphic, as a quandle, to a conjugacy class C of a group G so that $G = \langle C \rangle$ and $Z(G) = 1$.

Proof. (i) Observe that for every $x, y, z \in X$ the equality $\phi_x(y \triangleright z) = \phi_x(y) \triangleright \phi_x(z)$ holds by axiom (A1). In other words $\phi_x\phi_y = \phi_{x \triangleright y}\phi_x$ holds for any two elements $x, y \in X$. However, this means the map Φ takes $x \triangleright y$ into $\phi_x\phi_y\phi_x^{-1} = \phi_x \triangleright \phi_y$ and so Φ is a rack morphism between X and $\Phi(X)$. Further if Φ is injective then X and $\Phi(X)$ would be isomorphic quandles.

(ii) Suppose X is faithful. Pick an automorphism $\sigma \in \mathrm{Inn}(X)$. Since, for any $x \in X$,

$$\Phi(\sigma \cdot x) = \Phi(\sigma(x)) = \phi_{\sigma(x)} = \sigma\phi_x\sigma^{-1} = \sigma \cdot \Phi(x),$$

we see that the map Φ defines an isomorphism between the actions of $\mathrm{Inn}(X)$ on X and on $\Phi(X)$. Now, if σ lies in the center of $\mathrm{Inn}(X)$, then $\phi_{\sigma(x)} = \sigma\phi_x\sigma^{-1} = \phi_x$ for every $x \in X$ implying σ is the identity map on X since the map $\Phi: x \mapsto \phi_x$ is a bijection.

(iii) Consider the map $\Phi: G \to \mathrm{Inn}(X)$ taking an element $g := y_j \dots y_2 y_1$ of G, written as the product of some elements $y_i \in X$ for $1 \leq i \leq j$, to an element ϕ_g of the inner automorphism group $\mathrm{Inn}(X)$, where ϕ_g is defined as the composition

$$\phi_g := \phi_{y_j} \dots \phi_{y_2}\phi_{y_1}.$$

Since, for any element $x \in X$ we have

$$gxg^{-1} = (y_j \triangleright \cdots (y_2 \triangleright (y_1 \triangleright x)) \dots) = \phi_{y_j} \dots \phi_{y_2}\phi_{y_1}(x),$$

the map Φ is well-defined and defines a surjective group homomorphism. Clearly, $g \in \mathsf{Ker}(\Phi)$ if and only if ϕ_g is the identity map which is the case if and only if $gxg^{-1} = x$ for every generator element $x \in X$.

(iv) Suppose X is faithful and connected. Since X is faithful, the rack X is isomorphic to the conjugation rack $\Phi(X)$ and since X is connected, conjugation rack $\Phi(X)$ is a conjugacy class of the inner automorphism group $\mathsf{Inn}(X)$. By the previous part the center of $\mathsf{Inn}(X)$ is trivial.

Conversely, suppose X is isomorphic to a conjugacy class C of a group G so that $G = \langle C \rangle$ and $Z(G) = 1$. By the previous part, the rack C is connected and faithful. Therefore X is faithful and connected as well. ■

Lemma 3 (see [8, Lemma 1.21])**.** *Let X and Y be two racks and let $f: X \to Y$ be a surjective rack homomorphism. Then, for any $y_1, y_2 \in Y$, the cardinalities of the fibers $f^{-1}(y_2)$ and $f^{-1}(y_1 \rhd y_2)$ are equal. In particular, every fiber of Y has the same cardinality if Y is connected.*

Proof. We show that there exist a bijective function $B: X \to X$ such that

$$B(f^{-1}(y_2)) \subseteq f^{-1}(y_1 \rhd y_2) \text{ and } B^{-1}(f^{-1}(y_1 \rhd y_2)) \subseteq f^{-1}(y_2).$$

Let $B(x) := a \rhd x$, where $f(a) = y_1$. Observe that $B^{-1} = \phi_a^{-1} \in \mathsf{Inn}(X)$. Suppose $x \in f^{-1}(y_2)$. Then $f(a \rhd x) = f(a) \rhd f(x) = y_1 \rhd y_2$. That is, $B(x) \in f^{-1}(y_1 \rhd y_2)$. Next, suppose $x \in f^{-1}(y_1 \rhd y_2)$. Then $f(B^{-1}(x)) = f\phi_a^{-1}(x) = \phi_{f(a)}^{-1} f(x) = \phi_{y_1}^{-1}(y_1 \rhd y_2) = y_2$. That is, $B^{-1}(x) \in f^{-1}(y_2)$. ■

Remark. Let X be a finite connected rack and take two elements x, y of X. Let λ_y^x be the orbit size of y and $\bar{\lambda}_y^x$ be the orbit size of ϕ_y under the actions of $\langle \phi_x \rangle$. By Lemma 3, Φ^{-1} yields a partition of X into a disjoint union of fibers of the same size. Observe that for any integer k the element $\phi_x^k(y)$ of X does not lie in the fiber $\Phi^{-1}(\phi_y)$ unless $\phi_x^k \phi_y \phi_x^{-k} = \phi_y$. Therefore $\bar{\lambda}_y^x$ divides λ_y^x.

Let X be a finite connected rack whose profile is $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$ and let k be an integer. For a permutation σ of X, the *k-part* of σ is the set of elements of X which appear in a cycle of length k in the permutation σ. And the *λ_s-part* of X is the multiset formed by the λ_s-parts of elements of $\Phi(X)$. For example, if X is the fourth quandle with 12 elements in Vendramin's list, using Table 1, we see that the 2-part of X is

$$\{5,9,6,10,7,11,8,12,1,9,2,10,3,11,4,12,1,5,2,6,3,7,4,8\}.$$

Lemma 4. *Let X be a finite connected rack with profile $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$. Then the number of occurrences of each element of X in the λ_s-part of X is same and equal to $a_s\lambda_s/f$ for each λ_s $(0 \le s \le t)$, where f is the cardinality of a fiber of Φ.*

Proof. Let σ be an element of $\mathsf{Inn}(X)$. Conjugating each element of $\Phi(X)$ with σ yields a permutation of $\Phi(X)$. Since $\sigma\phi_x\sigma^{-1} = \phi_{\sigma(x)}$, if $\sigma(u) = v$ for some distinct elements u and v of X then the number of occurrences of u and v in the λ_s-part of X would be the same. The

claim follows from the facts that X is connected and the cardinality of $\Phi(X)$ is $\frac{n}{f}$, where n is the number of elements in X. ■

Next, we shall review Conjecture 1 in group theoretical terms. Let X be a finite connected rack and fix an element $x \in X$. Let $F := \langle \phi_x \rangle$ be the subgroup of $G := \mathrm{Inn}(X)$ generated by ϕ_x and let H be a conjugate of the centralizer of ϕ_x in G. Since the subgroup H of G is the stabilizer of some point in $\Phi(X)$, the action of G on $\Phi(X)$ by conjugation is equivalent to the action of G on the coset space G/H by left multiplication. Let $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$ be the profile of $\Phi(X)$. Then, the cycle containing ϕ_y in the permutation ϕ_{ϕ_x} would correspond to the cycle

$$(\phi_y H, \phi_x \phi_y H, \dots, \phi_x^{\lambda_s - 1} \phi_y H).$$

Observe that, the subgroup $\langle \phi_x^{\lambda_s} \rangle$ of F is also a subgroup of $\phi_y H \phi_y^{-1}$.

Proposition 5. *Let X be a finite connected rack and x be an element of X. Let F be the subgroup of $\mathrm{Inn}(X)$ generated by ϕ_x and K be the centralizer of ϕ_x in $\mathrm{Inn}(X)$. Let $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$ be the profile of X. The following statements hold:*

(i) If F intersects one of the conjugates of K trivially, then λ_s divides λ_t for each $0 \le s \le t$.

(ii) Conversely, if λ_s divides λ_t for each $0 \le s \le t$ and X is faithful, then F intersects one of the conjugates of K trivially.

Proof. (i) Let k be an integer and $\bar{\lambda} = (\bar{\lambda}_0^{\bar{a}_0}, \bar{\lambda}_1^{\bar{a}_1}, \dots, \bar{\lambda}_{t'}^{\bar{a}_{t'}})$ be the profile of $\Phi(X)$. Let H be a conjugate of K such that the intersection of F with H is trivial. That means

$$(H, \phi_x H, \dots, \phi_x^{k-1} H)$$

is an orbit under the action of F if and only if k equals to the order of ϕ_x. Then, clearly, k equals to $\bar{\lambda}_{t'}$. From the remark following Lemma 3 we see that k divides λ_s for some $0 \le s \le t$ so that the order of ϕ_x is equal to λ_t.

(ii) Since λ_s divides λ_t for each $0 \le s \le t$, we see that the order of ϕ_x is λ_t. By Proposition 2 we know that X and $\Phi(X)$ have the same profiles. Therefore, the order of ϕ_{ϕ_x} is λ_t as well. From this we conclude that there exists an element ϕ_y of $\Phi(X)$ so that $\phi_y K$ lies in an orbit of size λ_t under the action of F, which means the intersection of F with the centralizer of $\phi_{\phi_y(x)}$ is trivial. ■

3. Main Results

Let G be a group acting transitively on a set Ω. A partition of Ω form a block system if the following property is satisfied: for any element g of G an element of the partition is either mapped to itself by g or to another element of the partition. In that case we call the elements of the partition *blocks*. A block system is trivial if either the partition consists of the whole set or it consists of the singletons. We say G acts *primitively* on Ω if the only block systems are the trivial block systems.

Let X be a finite connected rack with profile $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$ and k be an integer. For a permutation σ of X, the $\tilde{k}$-*part* of σ is the set of elements of X which appear in a cycle whose length is a divisor of k in the permutation σ. Let x, y be two elements of X. If an automorphism σ of X centralizes ϕ_y, then as a permutation σ preserves λ_s-part of ϕ_y for each λ_s $(0 \leq s \leq t)$. Let K be the $\tilde{k}$-part of ϕ_x and $L := X \backslash K$ be the complement of K in X. If ϕ_x^k centralizes ϕ_y, then ϕ_y centralizes ϕ_x^k as well. However, that means as a permutation ϕ_y preserves the set K as well as the set L. Consider again the fourth quandle with 12 elements in Vendramin's list (see Table 1). Its profile is $\lambda = (1^1, 2^1, 3^1, 6^1)$ so that $\tilde{2}$-part of ϕ_i has 3 elements whereas $\tilde{3}$-part has 4 elements. Now, if ϕ_j centralizes ϕ_i^2 then $\tilde{2}$-parts of ϕ_i and ϕ_j must be the same since the cycle type of ϕ_i^2 is $(1^3, 3^3)$. Moreover, for this example the set of $\tilde{2}$-parts of permutations ϕ_j form a block system for the action of the inner automorphism group. However, we don't know any general method for relating $\tilde{k}$-parts with a block system.

Let H be the stabilizer of an element $x \in \Omega$. Then, the action of G on Ω is primitive if and only if the stabilizer H is a maximal subgroup of G. To see this, observe that if there exists a proper subgroup K of G that contains H strictly, then the cosets of K would be unions of some of the cosets of H in G. However, this determines a non-trivial block system for G/H under the action of G by left multiplication.

Theorem 6. *Let X be a finite connected rack with profile $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$. If the inner automorphism group $\mathsf{Inn}(X)$ acts primitively on X, then λ_s divides λ_t for each $0 \leq s \leq t$.*

Proof. By assumption $\mathsf{Inn}(X)$ acts primitively on X. Since the fibers of $\Phi: X \to \Phi(X)$ form a partition of X which is a block system for the action of $\mathsf{Inn}(X)$, the racks X and $\Phi(X)$ are isomorphic and have the same profiles. Moreover, by Proposition 2 the center of $\mathsf{Inn}(X)$ is trivial.

To prove the Theorem suppose contrarily λ_s does not divide λ_t for some $0 \leq s \leq t$. We want to derive a contradiction. Let us fix an element $x \in X$ and let y be an element of X different from x. Let $F := \langle \phi_x \rangle$ and $F' := \langle \phi_y \rangle$. Also, let H be the centralizer of F in $G := \mathsf{Inn}(X)$ and H' be the centralizer of F'. Notice that H' is a conjugate of H. Consider the intersection $I := F \cap H'$. Using Proposition 5(i) contrapositively we see that I is a non-trivial subgroup of G. Now I centralizes both F' and H, so it lies in the center of $\langle F', H \rangle$. Since G acts on X primitively by assumption, we see that H is a maximal subgroup of G. Clearly, we can choose y so that F' is not contained in H. However that means I lies in the center of G which is a contradiction. ■

Recall that in a symmetric group two elements are conjugate if and only if they share the same cycle type. Let C be a conjugacy class in a symmetric group S_d. Recall that a conjugacy class generates a normal subgroup of the group. Therefore, if $d \geq 5$, as a quandle the inner automorphism group $\mathsf{Inn}(C)$ of C is isomorphic to the symmetric group S_d when C is the conjugacy class of an odd permutation and it is isomorphic to the alternating group A_d when C is the conjugacy class of an even permutation different from the identity permutation. Observe that C is faithful when $d \geq 5$ by Proposition 2, since the centers of A_d and S_d are trivial.

Theorem 7. *Let C be a conjugacy class in a symmetric group S_d which is connected as a quandle. If $\lambda = (\lambda_0^{a_0}, \lambda_1^{a_1}, \dots, \lambda_t^{a_t})$ is the profile of C, then λ_s divides λ_t for each $0 \leq s \leq t$.*

Proof. It is clear that the Theorem holds when C is the conjugacy class of a central element. So we assume in the rest of the proof that C is the conjugacy class of a non-central element.

First, we prove the Theorem when the degree of S_d is equal to three or four. If $d = 3$, there are two possibilities. Either C is the conjugacy class of $(1,2)$ or it is the conjugacy class of $(1,2,3)$. In the first case, the inner automorphism group is S_3 and since the center of S_3 is trivial, C is faithful by Proposition 2. The Theorem holds in this case since the order of $(1,2)$ is a prime number. The latter possibility cannot occur as the conjugacy class of $(1,2,3)$ is not connected as a rack.

If $d = 4$, there are four possibilities. The conjugacy class of $(1,2)(3,4)$ generates an abelian subgroup so it is not connected. Also the conjugacy class of $(1,2,3)$ is not connected as a quandle since the action of A_4 on this set is not transitive. But in the remaining two cases the conjugacy class is connected as a quandle. Since the order of an element of the conjugacy class is a prime power in all those last two cases the Theorem holds when the degree of the symmetric group is four.

Next, suppose $d \geq 5$. From the previous explanations we know that C is faithful and the inner automorphism group is isomorphic to either A_d or S_d when $d \geq 5$. Now, fix an element x of C and let $\mu = (\mu_0^{b_0}, \mu_1^{b_1}, \dots, \mu_r^{b_r})$ be the cycle type of x in S_d. Moreover, since the map $\Phi: C \to \mathsf{Inn}(C)$ extends to a group isomorphism, using Proposition 5(i), it is enough to show that $\langle x \rangle$ intersects the centralizer of some element y of C trivially. Suppose $r \geq 1$. Let y be an element of C having the following property: For any $0 \leq s \leq r$ the μ_s-parts of x and y are not coinciding. If the value of $r \geq 1$, the existence of y in S_d or A_d is clear. Let ℓ be the order of x and k be an integer dividing ℓ. Obviously, x^k does not centralize y unless $k = \ell$. Hence the Theorem holds when C is the conjugacy class of an element containing at least two cycles of different lengths in its cycle decomposition. Next, suppose $r = 0$. In that case $d = b_0\mu_0$ and the order of x is μ_0. In turn, the order of ϕ_x is μ_0 as well and we may assume μ_0 is neither 1 nor a prime power since the Theorem holds trivially in those cases. Recall that the centralizer of x in S_d is isomorphic to $C_{\mu_0} \wr S_{b_0}$, where C_{μ_0} is the cyclic group of order μ_0. Moreover, for some conjugate y of x the intersection of $\langle x \rangle$ with the centralizer of y is trivial when $\mu_0 \geq 6$. This completes the proof. ■

The proof of Theorem 7 can be adapted to work when C is the conjugacy class of an alternating group. Notice that a conjugacy class of a simple group is necessarily connected as a quandle. The only complication arises when a conjugacy class of A_d is not a conjugacy class of S_d. Actually this may happen. Let x be an even permutation of S_n whose cycle type is $\mu = (\mu_0^{b_0}, \mu_1^{b_1}, \dots, \mu_r^{b_r})$. The conjugacy class of x in S_n splits into two different conjugacy classes in A_n if μ_s is an odd number and $b_s = 1$ for every $0 \leq s \leq r$. Otherwise, the conjugacy class of x in S_n is same with the conjugacy class of x in A_n. Even in the former case, fixing an element x of C, we can show that there exist an element y of C so that the centralizer of y intersects $\langle x \rangle$ trivially.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Acknowledgements

I would like to thank Volkmar Welker for suggesting that I work on Hayashi's Conjecture, David Stanovský for helping improve the statement and proof of Theorem 6, and the anonymous referee whose suggestions greatly improved the exposition. This study was supported by Scientific and Technological Research Council of Turkey (TUBITAK) under the Grant Number 122F490. The author thanks to TUBITAK for their supports.

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